TRINITY COLLEGE

Year 12 Mathematics Specialist 3,4
Test ~~2~~ 3 2021

Section 1 Calculator Free

Vectors in 3D

STUDENT'S NAME Solutions (PRESSER)

DATE: Wednesday 12 May **TIME:** 20 minutes **MARKS:** 20

INSTRUCTIONS:

Standard Items: Pens, pencils, drawing templates, eraser

Questions or parts of questions worth more than 2 marks require working to be shown to receive full marks.

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1. (3 marks)

Solve the system of equations

$$x+y+z=4$$
$$2x-y+z=0$$
$$3x+y+z=8$$

✓ eliminate 1 variable
✓ eliminate 2 variable
✓ correct soln

$$\sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & 4 \\ 2 & -1 & 1 & 0 \\ 3 & 1 & 1 & 8 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & 4 \\ 0 & 3 & 1 & 8 \\ 0 & 2 & 2 & 4 \end{array}\right] \begin{array}{l} \\ 2R_1-R_2 \\ 3R_1-R_3 \end{array}$$

$$\sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & 4 \\ 0 & 3 & 1 & 8 \\ 0 & 1 & 1 & 2 \end{array}\right] \begin{array}{l} \\ \\ \tfrac{1}{2}R_3 \end{array}$$

$$\sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & 4 \\ 0 & 3 & 1 & 8 \\ 0 & 2 & 0 & 6 \end{array}\right] \begin{array}{l} \\ \\ R_2-R_3 \end{array}$$

$\therefore \quad y=3$

Now $3y+z=8$

$\Rightarrow z=-1$

Now $x+y+z=4$

$\Rightarrow x=2$

2. (8 marks)

Consider the following three planes:

$$2x - y + 2z = 4 \quad — 1$$
$$x + y - 2z = 3 \quad — 2$$
$$x - 2y + kz = m \quad — 3$$

$$\left[\begin{array}{ccc|c} 2 & -1 & 2 & 4 \\ 1 & 1 & -2 & 3 \\ 1 & -2 & k & m \end{array}\right]$$

The system of equation has infinite solutions.

(a) Determine the values of k and m [3]

$$\sim \left[\begin{array}{ccc|c} 2 & -1 & 2 & 4 \\ 0 & -3 & 6 & -2 \\ 0 & 3 & 2-2k & 4-2m \end{array}\right] \begin{array}{l} \\ R_1 - 2R_2 \\ R_1 - 2R_3 \end{array}$$

✓ reason

✓ k

✓ m

Infinite solns → plane is a linear combination

$\Rightarrow (3) = (1) - (2)$

$\Rightarrow k = 4 \quad m = 1$

or

Infinite solns if

$2 - 2k = -6 \quad k = 4$

$4 - 2m = 2 \quad m = 1$

(b) Give a geometric interpretation of the solution for this system of equations for different values of m. [2]

✓ $m = 1$

✓ $m \neq 1$

If $m = 1$, planes intersect along a line

If $m \neq 1$, planes intersect one another forming a triangular prism

(c) Determine the vector equation of the line where the three planes intersect. [3]

let $z = \lambda$

✓ y

Now $-3y + 6\lambda = -2$

$\Rightarrow y = \frac{2}{3} - 2\lambda$

✓ x

Now $2x - y + 2z = 4$

$2x = 4 - 2\lambda + \frac{2}{3} + 2\lambda$

$x = \frac{14}{6}$

✓ vector eqn

$\therefore$ eqn of line is

$$r = \begin{pmatrix} 7/3 \\ 2/3 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 0 \\ -2 \\ 1 \end{pmatrix}$$

3. (9 marks)

Consider a particle whose position as a function of time is given by

$$\underset{\sim}{r}(t) = \begin{pmatrix} 2\sin(2t) \\ 2\cos(2t) \end{pmatrix}$$

(a) Prove the velocity of the particle is always tangential to its position vector. [3]

$$\underset{\sim}{v}(t) = \begin{pmatrix} 4\cos 2t \\ -4\sin 2t \end{pmatrix}$$

✓ $\underset{\sim}{v}(t)$

✓ dot

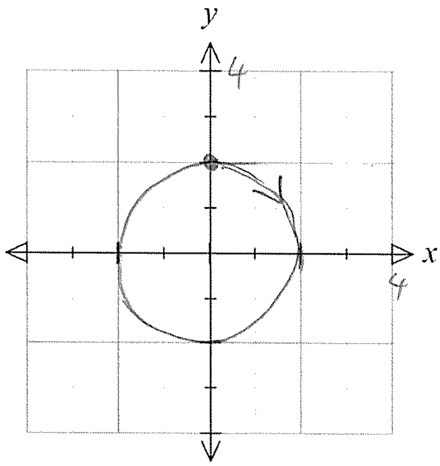
✓ expands

$$\text{So } \underset{\sim}{r}(t) \cdot \underset{\sim}{v}(t) = \begin{pmatrix} 2\sin 2t \\ 2\cos 2t \end{pmatrix} \cdot \begin{pmatrix} 4\cos 2t \\ -4\sin 2t \end{pmatrix}$$

$$= 8\sin 2t\cos 2t - 8\sin 2t\cos 2t$$

$$= 0$$

(b) Draw a sketch of the path of the particle and indicate the direction of motion. [3]



✓ Scale

✓ Circle

✓ direction

(c) Determine an expression for the total distance travelled by the particle between time a and time b. [3]

$$\text{dist} = \int_a^b |\underset{\sim}{v}(t)|\, dt$$

✓ expression

$$= \int_a^b \sqrt{(4\cos 2t)^2 + (-4\sin 2t)^2}\, dt$$

✓ magnitude

$$= \int_a^b 4\, dt$$

✓ simplified

$$= 4b - 4a$$

TRINITY COLLEGE

Year 12 Mathematics Specialist 3,4
Test 3 2021

Section 2 Calculator Assumed
Vectors in 3D

STUDENT'S NAME ________________________________

DATE: Wednesday 12 May **TIME:** 30 minutes **MARKS**: 31

INSTRUCTIONS:

Standard Items:	Pens, pencils, drawing templates, eraser
Special Items:	Three calculators, notes on one side of a single A4 page (these notes to be handed in with this assessment)

Questions or parts of questions worth more than 2 marks require working to be shown to receive full marks.

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4. (~~10~~ marks)

Two radio-controlled model planes take off at the same time from two different positions and with constant velocities. Model A leaves from the point with position vector $(-3\underset{\sim}{i} - 7\underset{\sim}{j})$ metres and has velocity $(5\underset{\sim}{i} - \underset{\sim}{j} + 2\underset{\sim}{k})$ m/s; model B leaves from the point with position vector $(7\underset{\sim}{i} - \underset{\sim}{j} - 8\underset{\sim}{k})$ metres and has velocity $(3\underset{\sim}{i} - 4\underset{\sim}{j} + 6\underset{\sim}{k})$ m/s.

(a) Determine the distance between the two model planes after 1 second of flight. [3]

$$\underset{\sim}{r}_A(1) = \begin{pmatrix} -3 \\ -7 \\ 0 \end{pmatrix} + \begin{pmatrix} 5 \\ -1 \\ 2 \end{pmatrix}$$

$$= \begin{pmatrix} 2 \\ -8 \\ 2 \end{pmatrix}$$

✓ $\underset{\sim}{r}_A(1)$

$$\underset{\sim}{r}_B(1) = \begin{pmatrix} 7 \\ -1 \\ -8 \end{pmatrix} + \begin{pmatrix} 3 \\ -4 \\ 6 \end{pmatrix}$$

$$= \begin{pmatrix} 10 \\ -5 \\ 2 \end{pmatrix}$$

✓ $\underset{\sim}{r}_B(1)$

$$\text{dist} = \left| \underset{\sim}{r}_a(1) - \underset{\sim}{r}_B(1) \right|$$

$$= \left| \begin{pmatrix} -8 \\ -3 \\ 4 \end{pmatrix} \right|$$

$$= \sqrt{89} \quad \text{or} \quad 9.43 \text{ m}$$

✓ distance

(b) Determine:

(i) an expression, in term of t, for the displacement between model plane A and model plane b. [1]

$$\underset{\sim}{r}_A(t) - \underset{\sim}{r}_B(t)$$

$$= \begin{pmatrix} -10 \\ -6 \\ 8 \end{pmatrix} + t\begin{pmatrix} 2 \\ 3 \\ -4 \end{pmatrix}$$

(ii) the shortest distance between the two model planes. [3]

$$\text{dist} = \sqrt{(-10+2t)^2 + (-6+3t)^2 + (8-4t)^2} \quad ✓$$

Using CAS and f-min ✓

$$\text{min value} = \frac{30\sqrt{29}}{29} \quad \text{or} \quad 5.57 \quad ✓$$

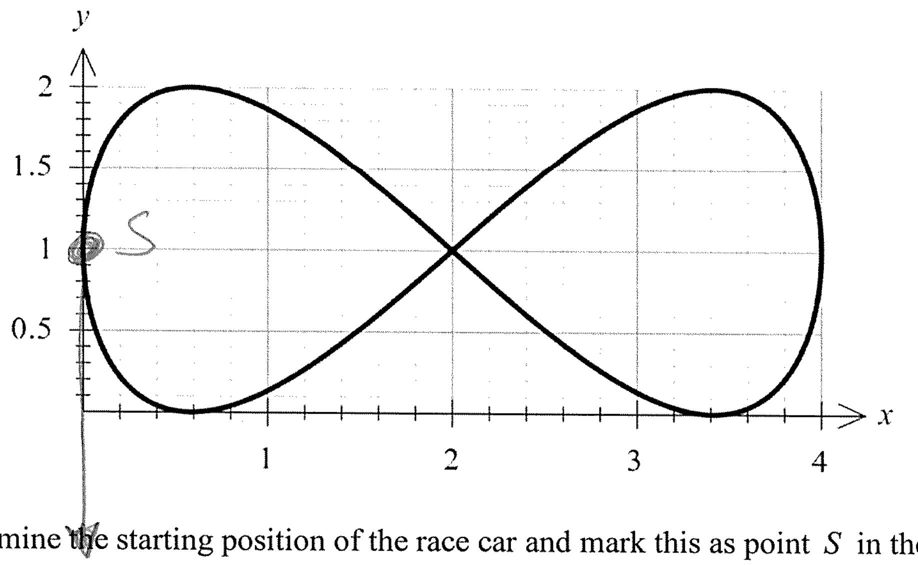
$$t = \frac{70}{29} \quad \text{or} \quad 2.41$$

(iii) the time when this occurs. [1]

After 2.41 seconds ✓

5. (10 marks)

The path of a toy race car on a racetrack is shown below. The race car moves so that its position vector $\underset{\sim}{r}(t)$ is given by $\underset{\sim}{r}(t) = \begin{pmatrix} 2-2\cos(t) \\ 1-\sin(2t) \end{pmatrix}$ metres, where t is the number of seconds the particle has been in motion.



(a) Determine the starting position of the race car and mark this as point S in the diagram above. [1]

$\underset{\sim}{r}(0) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

✓ S on diagram

(b) Determine the initial velocity of the race car and indicate this on the diagram above. [3]

$\underset{\sim}{v}(t) = \begin{pmatrix} 2\sin t \\ -2\cos 2t \end{pmatrix}$

✓ $\underset{\sim}{v}(t)$

✓ $\underset{\sim}{v}(0)$

$\underset{\sim}{v}(0) = \begin{pmatrix} 0 \\ -2 \end{pmatrix}$

✓ on graph

(c) Determine the Cartesian equation for the path of the race car. [3]

✓ expression $\cos t$

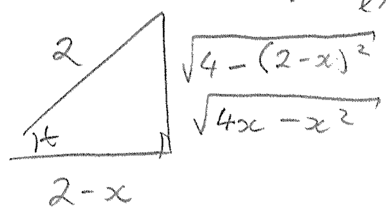
✓ sub for $\sin t$

✓ expression y^2

$$x = 2 - 2\cos t$$

$$\Rightarrow \frac{x-2}{-2} = \cos t$$

$$\Rightarrow \frac{2-x}{2} = \cos t$$



$$y = 1 - \sin 2t$$

$$= 1 - 2\sin t \cos t$$

$$= 1 - \cancel{2}\,\frac{\sqrt{4x - x^2}}{\cancel{2}} \cdot \frac{(2-x)}{2}$$

$$= 1 - \tfrac{1}{2}\sqrt{4x - x^2}\,(2-x)$$

But, this is only the top half ✓

$$\therefore y^2 = \left(1 - \tfrac{1}{2}\sqrt{4x - x^2}\,(2-x)\right)^2$$

(d) Determine the distance the race car travels in completing one circuit of the racetrack. [3]

Back at start when $x = 0$

$$\Rightarrow 0 = 2 - 2\cos t$$ ✓ $t = 2\pi$

$$\Rightarrow 1 = \cos t$$

$$\Rightarrow t = 0, 2\pi, 4\pi, \ldots$$

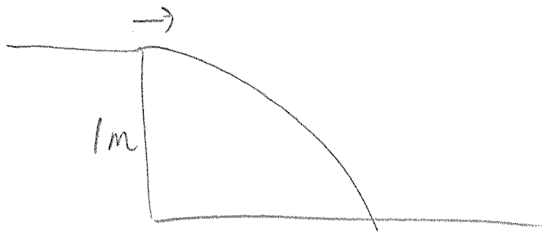
$$\therefore \text{dist} = \int_0^{2\pi} |\underline{v}(t)|\, dt$$ ✓ exp

$$= 12.19 \text{ m}$$ ✓ answer

6. (11 marks)

A ball rolls off a table with a speed of 60 cm/s. The table is 1 m high. The ball undergoes acceleration due to gravity of $\underset{\sim}{a}(t) = \begin{pmatrix} 0 \\ -9.8 \end{pmatrix}$ m/s^2

(a) Determine the point at which the ball hits the floor and determine the speed at the instant. [5]



$\underset{\sim}{v}(0) = \begin{pmatrix} 0.6 \\ 0 \end{pmatrix}$ cm/s

✓ $\underset{\sim}{v}(t)$
✓ $\underset{\sim}{r}(t)$
✓ t values
✓ $\underset{\sim}{r}\left(\frac{\sqrt{10}}{7}\right)$
✓ speed

$\underset{\sim}{a}(t) = \begin{pmatrix} 0 \\ -9.8 \end{pmatrix}$

$\underset{\sim}{v}(t) = \begin{pmatrix} 0 \\ -9.8t \end{pmatrix} + \underset{\sim}{c}$ but $\underset{\sim}{v}(0) = \begin{pmatrix} 0.6 \\ 0 \end{pmatrix}$

$\therefore \underset{\sim}{v}(t) = \begin{pmatrix} 0.6 \\ -9.8t \end{pmatrix}$

$\underset{\sim}{r}(t) = \begin{pmatrix} 0.6t \\ -4.9t^2 \end{pmatrix} + \underset{\sim}{d}$ but $\underset{\sim}{r}(0) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$\therefore \underset{\sim}{r}(t) = \begin{pmatrix} 0.6t \\ -4.9t^2 + 1 \end{pmatrix}$

Hit ground when $-4.9t^2 + 1 = 0$

$\Rightarrow t = \pm \frac{\sqrt{10}}{7}$, so $\frac{\sqrt{10}}{7}$

$\underset{\sim}{r}\left(\frac{\sqrt{10}}{7}\right) = \begin{pmatrix} \frac{3\sqrt{10}}{35} \\ 0 \end{pmatrix} = \begin{pmatrix} 0.271 \\ 0 \end{pmatrix}$ m

speed impact $\left|\underset{\sim}{v}\left(\frac{\sqrt{10}}{7}\right)\right| = \frac{\sqrt{499}}{5} = 4.47$ cm/s

(b) Determine the angle θ between the path of the ball and a vertical line drawn through the point of impact. [3]

$$\text{angle}\left(\underset{\sim}{v}\left(\frac{\sqrt{10}}{7}\right), \begin{pmatrix}0\\1\end{pmatrix}\right)$$ ✓ $\underset{\sim}{v}\left(\frac{\sqrt{10}}{7}\right)$

$$= \text{angle}\left(\begin{pmatrix}3/5\\ -\frac{7\sqrt{10}}{5}\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix}\right)$$ ✓ 172.3°

✓ 7.3°

$$= 172.28^\circ$$

$\therefore$ angle is 7.72°

(c) Suppose the ball rebounds from the floor at the same angle with which it hits the floor but loses 20% of its speed due to energy absorbed by the ball on impact. Where does the ball strike the floor on the second bounce? [3]

new speed 80% of $\frac{\sqrt{499}}{5} = \frac{4\sqrt{499}}{25}$

assuming new coordinate system yields

$$\underset{\sim}{r}(t) = \begin{pmatrix} \frac{4\sqrt{499}}{25}\sin(7.72^\circ)t \\ \frac{4\sqrt{499}}{25}\cos(7.72^\circ)t - 4.9t^2 \end{pmatrix}$$ ✓ $\underset{\sim}{r}(t)$

solving when y-comp is 0 gives

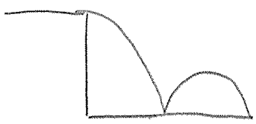
$t = 0,\ 0.7228$ ✓ time

$\therefore$ range $= \frac{4\sqrt{499}}{25}\sin(7.72^\circ) \times 0.7228$

$= 0.347$ m ✓ location

$\therefore$ $0.271 + 0.347 = 0.618$ m left of table

$\frac{0}{2}$



0.27

If lose 20% → keep 80%

but 80% in 2D

$\therefore$ 2nd bounce $= 0.27 + 0.8^2 \times 0.27 \times 2$